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Generalized Solutions to Burgers' Equation

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INTRODUCTION

In this article we study generalized solutions in the sense of Colombeau [4, 5] to the Cauchy problem

$$\begin{aligned} u_t + uu_x &= \nu u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0 \\ u|_{t=0} &= u_0, \qquad x \in \mathbb{R}, \end{aligned} \tag{1}$$

where ν is a positive constant or generalized constant. The solutions will belong to an algebra $\mathscr{G}_{s,g}$ of generalized functions, to be defined below, which contains the space of bounded distributions $\mathscr{D}'_{L^\infty}$. In particular, the initial data may be arbitrary bounded distributions.

Before we go on describing our results, let us explain the setting we use. Our aim is to look for solutions in large differential algebras of generalized functions, so that all differentiations and nonlinear operations involved can be performed unrestrictedly. Following Colombeau [4, 5] and also Rosinger [15], we construct these algebras by putting an algebraic structure on certain spaces made up by nets of smooth functions on open or closed subsets of $\mathbb{R}^n$. In order not to complicate matters, we shall take all of $\mathbb{R}^n$ as the underlying domain for the purpose of the introduction.

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The starting point is an infinite product of the space $C^\infty(\mathbb{R}^n)$, which we denote by $\mathscr{E}_s[\mathbb{R}^n]$ in accordance with [1],

$$\mathscr{E}_s[\mathbb{R}^n] = \{f: (0, \infty) \times \mathbb{R}^n \to \mathbb{C}: (\varepsilon, x) \to f(\varepsilon, x), \text{ such that } f \text{ is } C^\infty \text{ in the variable } x \in \mathbb{R}^n \text{ for each } \varepsilon \in (0, \infty)\}.$$

It is clear that $\mathscr{E}_s[\mathbb{R}^n]$ is an algebra with partial derivatives, the operations being performed with respect to the variable x at each fixed ε. Every net of smooth functions with compact support $(\varphi_\varepsilon)_{\varepsilon>0}$ converging to the Dirac measure in the space of distributions $\mathscr{D}'(\mathbb{R}^n)$ gives rise to an imbedding of that space by

$$u \to [(\varepsilon, x) \to (u * \varphi_\varepsilon)(x)]$$

for $u \in \mathscr{D}'(\mathbb{R}^n)$. This imbedding is linear and preserves derivatives, but does not preserve any nonlinear operations, in particular, not even the multiplication on the subspace $C^\infty(\mathbb{R}^n)$ of $\mathscr{D}'(\mathbb{R}^n)$. That is why we have to single out a certain subalgebra of $\mathscr{E}_s[\mathbb{R}^n]$, denoted by $\mathscr{E}_{M,s}[\mathbb{R}^n]$, and go over to the quotient algebra with respect to a suitable ideal $\mathscr{N}_s(\mathbb{R}^n)$. Thus our generalized functions will be elements of a quotient space, namely $\mathscr{G}_s(\mathbb{R}^n) = \mathscr{E}_{M,s}[\mathbb{R}^n]/\mathscr{N}_s(\mathbb{R}^n)$. Colombeau has shown that it is possible to choose $\mathscr{E}_{M,s}[\mathbb{R}^n]$ and $\mathscr{N}_s(\mathbb{R}^n)$ in such a way that $\mathscr{G}_s(\mathbb{R}^n)$ still contains $\mathscr{D}'(\mathbb{R}^n)$, while it is an algebra with derivatives extending the distributional ones, and $C^\infty(\mathbb{R}^n)$ is a faithful subalgebra. In view of certain impossibility results, discussed in detail in [15], these properties together are the best one can achieve when one imbeds $\mathscr{D}'(\mathbb{R}^n)$ into algebras. In addition, superposition of elements of $\mathscr{G}_s(\mathbb{R}^n)$ by smooth functions of polynomial growth is possible, as is restriction of its elements to subspaces of $\mathbb{R}^n$. Thus nonlinear Cauchy problems, like problem (1), can be formulated in such a setting. We shall follow Colombeau's approach, but in our investigation of problem (1) we found it necessary to employ a certain modified version $\mathscr{G}_{s,g}(\mathbb{R}^n)$ with stronger boundedness properties built in.

Let us point out here that the space of distributions $\mathscr{D}'(\mathbb{R}^n)$ itself can be recovered by means of a similar quotient construction: let $\mathscr{W}(\mathbb{R}^n)$ be the linear subspace of $\mathscr{E}_s[\mathbb{R}^n]$ consisting of those f for which $f(\varepsilon, \cdot)$ converges in $\mathscr{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$, and let $\mathscr{V}(\mathbb{R}^n) \subset \mathscr{W}(\mathbb{R}^n)$ be the subspace of those f for which $f(\varepsilon, \cdot) \to 0$ as $\varepsilon \to 0$ in $\mathscr{D}'(\mathbb{R}^n)$. Then $\mathscr{D}'(\mathbb{R}^n)$ is linearly isomorphic with the quotient $\mathscr{W}(\mathbb{R}^n)/\mathscr{V}(\mathbb{R}^n)$. It is worthwhile to note that neither $\mathscr{W}(\mathbb{R}^n)$ nor $\mathscr{V}(\mathbb{R}^n)$ are algebras; in fact, no proper ideal of any subalgebra of $\mathscr{E}_s[\mathbb{R}^n]$ can contain $\mathscr{V}(\mathbb{R}^n)$. It turns out that the ideal $\mathscr{N}_s(\mathbb{R}^n)$ is a proper subspace of $\mathscr{V}(\mathbb{R}^n)$. In this sense the quotient structure of the algebra $\mathscr{G}_s = \mathscr{E}_{M,s}/\mathscr{N}_s$ is much finer than the one of $\mathscr{D}' = \mathscr{W}/\mathscr{V}$. For instance, the powers of the Heaviside function are all different as elements of $\mathscr{G}_s(\mathbb{R})$, while they are equal when viewed as elements of $\mathscr{D}'(\mathbb{R})$; and the same actually happens in every commutative differential algebra containing the distributions.

We now come to a delicate point. The observation above, concerning powers of classical functions, shows that the information carried by the elements of $\mathscr{G}_s = \mathscr{E}_{M,s}/\mathscr{N}_s$ is appreciably larger than the one carried by the objects of distribution theory. So, to obtain consistency with certain classically definable nonlinear operations, some device is needed to bring this information down to the usual level. This is achieved by means of the concept of association, introduced by Colombeau [4, 5]: two elements f, g of $\mathscr{G}_s(\mathbb{R}^n) = \mathscr{E}_{M,s}[\mathbb{R}^n]/\mathscr{N}_s(\mathbb{R}^n)$ are called associated, denoted by $f \approx g$, iff their difference belongs to $\mathscr{V}(\mathbb{R}^n)$, that is, iff the difference of any two representatives in $\mathscr{E}_s[\mathbb{R}^n]$ converges to zero in the sense of distributions. As an example, we now have that the powers of the Heaviside function are associated with each other. We shall also see below that it will become necessary to replace equality by association in the formulation of the inviscid Burgers' equation.

With the basic notions explained, we can now say what we shall do in this article. First, we establish existence and uniqueness of a solution to problem (1), where ν is allowed to be an arbitrary generalized constant (this includes the classical viscous Burgers' equation when ν is actually a positive real number). Second, we let ν be associated with zero (this means that ν is the class of a sequence of constants $\hat{\nu}(\varepsilon)$ converging to zero as $\varepsilon \to 0$). Then, if u is a generalized solution to (1) given by a family $\hat{u}(\varepsilon, \cdot)$ of smooth functions which is bounded independently of ε, we have that $\hat{\nu}(\varepsilon)\, \hat{u}(\varepsilon, \cdot)_{xx} \to 0$ in $\mathscr{D}'(\mathbb{R} \times (0, \infty))$. It follows that this same generalized function u satisfies the inviscid Burgers' equation

$$\begin{aligned} u_t + u u_x &\approx 0 \\ u|_{t=0} &= u_0 \end{aligned} \tag{2}$$

written with association. This way we obtain a method for solving the Cauchy problem (2) with arbitrary initial data. The question of uniqueness of generalized solutions to (2) will be addressed below; let us point out here that the formulation (2) of the inviscid Burgers' equation is the correct one in the setting of Colombeau's theory. In fact, the stronger formulation

$$\begin{aligned} u_t + u u_x &= 0 \\ u|_{t=0} &= u_0 \end{aligned} \tag{3}$$

with equality understood in the sense of the Colombeau algebras of generalized functions $\mathscr{G}_s$ or $\mathscr{G}_{s,g}$ is not suitable as it does not admit shock wave solutions.[1] On the other hand, (2) has shock wave solutions in $\mathscr{G}_s$ or $\mathscr{G}_{s,g}$, which, in addition, necessarily satisfy the classical Rankine–Hugoniot condition. The crucial observation is that one can solve the general Cauchy problem for (1) as well as for (2) in the setting of Colombeau algebras, but not for (3).

Explicit generalized shock wave solutions for conservative and non-conservative hyperbolic systems have been constructed in [1, 3, 7, 8]. The present article is part of a program to obtain generalized solutions to hyperbolic conservation laws by adding a viscous or dispersive term which is associated with zero, see our forthcoming paper [2].

The plan of exposition is as follows. In Section 1 we recall the definitions of the Colombeau algebras $\mathscr{G}_s(\Omega)$ and $\mathscr{G}_s(\bar{\Omega})$, and introduce the modified version $\mathscr{G}_{s,g}(\bar{\Omega})$, where the local bounds defining the elements of $\mathscr{G}_s(\bar{\Omega})$ are replaced by global bounds. This modification is necessary in order to retain uniqueness of solutions to (1); an example of nonuniqueness, when the space $\mathscr{G}_s(\mathbb{R}\times[0,\infty))$ is used, is given in Section 4. Moreover, growth conditions on the initial data are needed in order to avoid blow up in finite time, as has been observed by Hopf [10] already. While $\mathscr{G}_s(\mathbb{R}^n)$ contains $\mathscr{D}'(\mathbb{R}^n)$ and has $C^\infty(\mathbb{R}^n)$ as a subalgebra, $\mathscr{G}_{s,g}(\mathbb{R}^n)$ contains $\mathscr{D}'_{L^\infty}(\mathbb{R}^n)$, the space of distributional derivatives of bounded functions, and it has $\mathscr{D}_{L^\infty}(\mathbb{R}^n)$ as a subalgebra, the space of smooth functions with bounded derivatives.

In Section 2 we establish existence of a solution $u\in\mathscr{G}_{s,g}(\mathbb{R}\times[0,\infty))$ to problem (1), given arbitrary initial data $u_0\in\mathscr{G}_{s,g}(\mathbb{R})$. Then we give three sufficient conditions yielding uniqueness: boundedness of u, nonnegativity of $\partial_x u$, or logarithmic dependence on the regularization parameter for $\partial_x u$. Under the latter hypotheses, the solution is shown to depend continuously on the initial data with respect to a suitable Hausdorff topology on $\mathscr{G}_{s,g}$.

Section 3 is devoted to studying how the generalized solutions to (1) relate to the classical solutions, when the latter exist. In the case where the viscosity coefficient ν is associated with zero and the initial data belong to $L^\infty(\mathbb{R})$, it turns out that the generalized solution to (1) is associated with the classical weak solution to (3) which satisfies the entropy condition. This follows from an argument of Lax [12] employing the weak norm $|\cdot|_*$ on L^∞. We present a slight generalization thereof, which is needed subsequently. We also calculate the associated distribution when u_0 is associated

[1] In fact, if u is a piecewise smooth function which satisfies Eq. (3) as an element of $\mathscr{G}_s$ or $\mathscr{G}_{s,g}$, then necessarily the classical Rankine–Hugoniot jump condition holds for u. But multiplication of (3) by u, which is meaningful in the setting of a differential algebra, turns (3) into the conservation law $uu_t + u^2u_x = 0$, which is known to imply different jump conditions than (3). The same solution u cannot satisfy contradicting jump conditions, thus (3) does not have shock wave solutions in $\mathscr{G}_s$ or $\mathscr{G}_{s,g}$. Equation (3) allows for rarefaction wave solutions, though, as indicated in Section 4.

with the Dirac measure. The result is not surprising—it coincides with the heuristically derived solution as given, e.g., in [16].

Next, when ν is a real constant and the initial data are bounded, the generalized solution is associated with the classical one as well. If the initial data belong to $\mathscr{D}_{L^\infty}(\mathbb{R})$, the generalized solution is actually equal to the classical one in $\mathscr{G}_{s,g}(\mathbb{R}\times[0,\infty))$. Finally, we also calculate the associated distribution when ν is associated with ∞.

In the last section we address the question of uniqueness of solutions to Eq. (2); as mentioned before, existence of a solution follows from the results of Section 2. First, since every piecewise continuous classical weak solution to (3) solves (2), when viewed as an element of $\mathscr{G}_{s,g}(\mathbb{R}\times[0,\infty))$, we must have at least the degree of nonuniqueness present already in the classical case. Let us call a solution $u\in\mathscr{G}_{s,g}(\mathbb{R}\times[0,\infty))$ to (2) a *zero viscosity limit*, if u also solves (1) for some generalized constant ν such that both ν and νu_{xx} are associated with zero. Clearly, two zero viscosity limit solutions to (2) which arise from the same generalized constant ν are equal, due to the uniqueness theory of Section 2. So the question is as follows: given two zero viscosity limit solutions u_1 and u_2 to (2) which arise as solutions to (1) with differing viscosity coefficients ν_1 and ν_2, respectively, what is their relation? We can show that if ν_1/ν_2 is associated with 1 and u_1, u_2 satisfy certain boundedness assumptions, then u_1 is associated with u_2.

Finally, we discuss the question of uniqueness (up to association) for the problem

$$\begin{aligned} u_t + uu_x &\approx 0 \\ u|_{t=0} &\approx u_0. \end{aligned} \tag{4}$$

Since Eq. (1) has soliton solutions which travel with arbitrarily large speed and originate arbirtrarily close to $-\infty$ at time zero, it is easy to construct a solution to (4) which is associated with zero initially, but not associated with any distribution for positive times. Thus two solutions to (4) need not be associated even if they are zero viscosity limits. However, we can show that two solutions to (4) which arise as zero viscosity limits (again with $\nu_1/\nu_2\approx 1$) are associated with each other, provided the initial data have compact support and satisfy a boundedness assumption.

We remark that the same example demonstrating nonuniqueness for (4) also shows that the solutions to (1) are not unique when equality is understood in the sense of $\mathscr{G}_s$ rather than $\mathscr{G}_{s,g}$, as mentioned earlier.

As a byproduct of the results in this section we give an example of a solution in $\mathscr{G}_{s,g}(\mathbb{R}\times[0,\infty))$ to (3) which is not a classical smooth function (rather, it is a rarefaction wave). Other examples involving quasilinear systems have recently been given by Colombeau and Heibig in [6].

We should like to elaborate a bit more on the uniqueness issue. In the classical case, the entropy condition in various forms [12, 14] singles out a unique weak solution to (3), which coincides with the one obtained by the vanishing viscosity method. In as much as our solutions are obtained as zero viscosity limits, they satisfy the entropy condition in the sense that they are associated with the classical entropy solution, as we show in Section 3. One understands the lack or presence of uniqueness better if one notes that it actually amounts to a lack or presence of stability. In fact, uniqueness of solutions to (4) means the following: if $(u-v)|_{t=0} \in \mathscr{V}(\mathbb{R})$, the space of distributional zero sequences, and if both $u_t + uu_x$ and $v_t + vv_x$ belong to $\mathscr{V}(\mathbb{R} \times (0, \infty))$, then $u - v \in \mathscr{V}(\mathbb{R} \times (0, \infty))$ as well. Or, put in a more explicit fashion, if the representatives $\hat{u}(\varepsilon, \cdot)$ of u and $\hat{v}(\varepsilon, \cdot)$ of v have the property that $(\hat{u}(\varepsilon, \cdot) - \hat{v}(\varepsilon, \cdot))|_{t=0} \to 0$ in $\mathscr{D}'(\mathbb{R})$, $\hat{u}(\varepsilon, \cdot)_t + \hat{u}(\varepsilon, \cdot)\, \hat{u}(\varepsilon, \cdot)_x \to 0$, and $\hat{v}(\varepsilon, \cdot)_t + \hat{v}(\varepsilon, \cdot)\, \hat{v}(\varepsilon, \cdot)_x \to 0$ in $\mathscr{D}'$ $(\mathbb{R} \times (0, \infty))$, then $\hat{u}(\varepsilon, \cdot) - \hat{v}(\varepsilon, \cdot) \to 0$ in $\mathscr{D}'(\mathbb{R} \times (0, \infty))$. This obviously is a stability property (to be sure, it holds only under the additional assumptions outlined in Section 4). Uniqueness of generalized solutions to problem (1) can be viewed in the same light, the role of the space $\mathscr{V}$ being taken by the ideal $\mathscr{N}_s$. Thus we are led to observe that the occurrence of uniqueness is a special type of stability. As this point of view is not so common, we should like to emphasize that the same coincidence happens in the setting of distributional solutions to linear equations. Indeed, let $P(D)$ be a linear partial differential operator. Unique solvability in $\mathscr{D}'$ would say that if $u, v \in \mathscr{D}'$, $P(D)\, u = P(D)\, v$ in $\mathscr{D}'$, then $u = v$. Interpreting $\mathscr{D}'$ as the quotient space $\mathscr{W}/\mathscr{V}$, unique solvability means stability with respect to distributional convergence: if $\hat{u}$, $\hat{v} \in \mathscr{W}$ are some representatives of u, v in $\mathscr{W}/\mathscr{V}$, then $P(D)\, u = 0$ says that $P(D)\, \hat{u} \in \mathscr{V}$; unique solvability asserts that $P(D)\, \hat{u} - P(D)\, \hat{v} \in \mathscr{V}$ implies $\hat{u} - \hat{v} \in \mathscr{V}$.

1. The Algebras of Generalized Functions

We give here the simplified definitions of the Colombeau algebras of generalized functions, $\mathscr{G}_s(\Omega)$, $\mathscr{G}_s(\bar{\Omega})$, and $\mathscr{G}_{s,g}(\bar{\Omega})$, with Ω a nonvoid open subset of $\mathbb{R}^n$, and $\bar{\Omega}$ the closure of Ω.

1.1. *Notations.* We set

$$\mathscr{E}_s[\Omega] = \{f\colon (0, \infty) \times \Omega \to \mathbb{C} \text{ such that } f \text{ is } C^\infty \text{ in the variable } x \in \Omega, \text{ for each } \varepsilon > 0\};$$

$$\mathscr{E}_{M,s}[\Omega] = \{ f \in \mathscr{E}_s[\Omega] \text{ such that for all compact subsets } K \text{ of } \Omega \text{ and } \alpha \in \mathbb{N}^n \text{ there is } N \in \mathbb{N}, \text{ such that } \sup_{x \in K} |\partial_x^\alpha f(\varepsilon, x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\};$$

$$\mathscr{N}_s(\Omega) = \{f \in \mathscr{E}_{M,s}[\Omega] \text{ such that for all } K \text{ and } \alpha \text{ as above and all } q \in \mathbb{N}, \sup_{x \in K} |\partial_x^\alpha f(\varepsilon, x)| = O(\varepsilon^q) \text{ as } \varepsilon \to 0\};$$

finally we set

$$\mathscr{G}_s(\Omega) = \mathscr{E}_{M,s}[\Omega]/\mathscr{N}_s(\Omega).$$

The algebra $\mathscr{G}_s(\bar{\Omega})$ is defined analogously, except that $\mathscr{E}_{M,s}[\bar{\Omega}]$ is a subalgebra of

$$\mathscr{E}_s[\bar{\Omega}] = \{f\colon (0, +\infty) \times \bar{\Omega} \to \mathbb{C} \text{ such that } f|_{(0,\infty)\times\Omega} \in \mathscr{E}_s[\Omega] \text{ and the map } x \in \Omega \to f(\varepsilon, x) \in \mathbb{C} \text{ and all its derivatives can be continuously extended to } \bar{\Omega}, \text{ for each } \varepsilon > 0\};$$

1.2. DEFINITIONS. (a) We say that a generalized function on Ω is *real valued* (*positive*) if it has a real valued (positive) representative.

(b) A generalized function $u \in \mathscr{G}_s(\Omega)$ is *associated with a distribution* w if it has a representative $\hat{u} \in \mathscr{E}_{M,s}[\Omega]$ such that

$$\hat{u}(\varepsilon, \cdot) \to w \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad \mathscr{D}'(\Omega).$$

Notation: $u \approx w$.

(c) Two generalized functions $u_1, u_2 \in \mathscr{G}_s(\Omega)$ are said to be *associated with each other* if $u_1 - u_2 \approx 0$. Notation: $u_1 \approx u_2$.

(d) A generalized function $u \in \mathscr{G}_s(\Omega)$ is called a *generalized constant* if it has a representative which is constant for each $\varepsilon > 0$.

1.3. *Notations.* We set

$$\mathscr{E}_{M,s,g}[\bar{\Omega}] = \{f \in \mathscr{E}_s[\bar{\Omega}] \text{ such that for all } \alpha \in \mathbb{N}^n \text{ there is } N \in \mathbb{N} \text{ such that } \sup_{x\in\Omega} |\partial_x^\alpha f(\varepsilon, x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}; \tag{5}$$

$$\mathscr{N}_{s,g}(\bar{\Omega}) = \{f \in \mathscr{E}_{M,s,g}[\bar{\Omega}] \text{ such that for all } \alpha \in \mathbb{N}^n \text{ and } q \in \mathbb{N}\ \sup_{x\in\Omega} |\partial_x^\alpha f(\varepsilon, x)| = O(\varepsilon^q) \text{ as } \varepsilon \to 0\}; \tag{6}$$

finally we set

$$\mathscr{G}_{s,g}[\bar{\Omega}] = \mathscr{E}_{M,s,g}[\bar{\Omega}]/\mathscr{N}_{s,g}(\bar{\Omega}),$$

where the subscript "g" stands for "globally," due to the fact that all estimates are taken globally.

1.4. *Remark.* In analogy to the imbedding of $\mathscr{D}'(\mathbb{R}^n)$ into $\mathscr{G}_s(\mathbb{R}^n)$ which renders $C^\infty(\mathbb{R}^n)$ a subalgebra, we can construct an imbedding from $D'_{L^\infty}(\mathbb{R}^n)$, the space of distributional derivatives of bounded functions, into

$\mathcal{G}_{s,g}(\mathbb{R}^n)$, which renders $\mathcal{D}_{L^\infty}(\mathbb{R}^n)$, the space of smooth functions with bounded derivatives, a subalgebra. Such an imbedding is given by

$$(\iota_\rho w)(\varepsilon, x) = (w * \rho_\varepsilon)(x)$$

for $w \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$, $\varepsilon > 0$, $x \in \mathbb{R}^n$, where ρ is a fixed element of $\mathcal{S}(\mathbb{R}^n)$ satisfying $\int \rho(x)\,dx = 1$, $\int x^\alpha \rho(x)\,dx = 0$, for all $\alpha \in \mathbb{N}^n$, $|\alpha| \geqslant 1$, and

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\,\rho\left(\frac{x}{\varepsilon}\right).$$

1.5. Definition. We define a topology on $\mathcal{G}_{s,g}(\bar{\Omega})$ as follows: if $p, q \in \mathbb{N}$, we denote by $V(p, q)$ the set of all $u \in \mathcal{G}_{s,g}(\bar{\Omega})$ which have a representative $\hat{u}$ such that

$$\sup_{x \in \bar{\Omega}, |\alpha| \leqslant p} |\partial_x^\alpha \hat{u}(\varepsilon, x)| \leqslant \varepsilon^q$$

for $\varepsilon > 0$ small enough. These sets are a basis of 0-neighbourhoods on $\mathcal{G}_{s,g}(\bar{\Omega})$ and, from the definition of $\mathcal{N}_{s,g}(\bar{\Omega})$, this topology is Hausdorff.

1.6. Definition. We say that $u \in \mathcal{G}_{s,g}(\bar{\Omega})$ is of *log-type* (resp. *bounded type*) if it has a representative $\hat{u} \in \mathcal{E}_{M,s,g}[\bar{\Omega}]$ such that

$$\sup_{x \in \bar{\Omega}} |\hat{u}(\varepsilon, x)| = O\left(\log\frac{1}{\varepsilon}\right) \qquad \text{(resp. } O(1)\text{) as } \varepsilon \to 0. \tag{7}$$

2. Existence and Uniqueness

2.1. Theorem. *Let v be a generalized constant with a representative $\hat{v}$ satisfying: there are $N \in \mathbb{N}$ and $\eta > 0$ such that for each $0 < \varepsilon < \eta$*

$$\hat{v}(\varepsilon) \geqslant \varepsilon^N. \tag{8}$$

Then, given $u_0 \in \mathcal{G}_{s,g}(\mathbb{R})$, there is a solution $u \in \mathcal{G}_{s,g}(\mathbb{R} \times [0, \infty))$ of (1).

Proof. Let, for $(x, t) \in \mathbb{R} \times (0, \infty)$ and $\varepsilon > 0$,

$$\hat{u}(\varepsilon, x, t) = \frac{\displaystyle\int_{-\infty}^{+\infty} \exp\left(\frac{-y^2}{4\hat{v}(\varepsilon)\,t}\right) \hat{u}_0(\varepsilon, x-y) \exp\left(-\frac{1}{2\hat{v}(\varepsilon)} \int_0^{x-y} \hat{u}_0(\varepsilon, \xi)\,d\xi\right) dy}{\displaystyle\int_{-\infty}^{+\infty} \exp\left(\frac{-y^2}{4\hat{v}(\varepsilon)\,t}\right) \exp\left(-\frac{1}{2\hat{v}(\varepsilon)} \int_0^{x-y} \hat{u}_0(\varepsilon, \xi)\,d\xi\right) dy},$$

where $\hat{u}_0 \in \mathcal{E}_{M,s,g}[\mathbb{R}]$ is a representative of u_0. We have that, for each $\varepsilon > 0$, $\hat{u}(\varepsilon, \cdot)$ is a solution of (1), and it is smooth up to $t = 0$ (see [10]), so

$\hat{u} \in \mathscr{E}_s[\mathbb{R} \times [0, \infty)]$. Let us prove that $\hat{u} \in \mathscr{E}_{M,s,g}[\mathbb{R} \times [0, \infty)]$. For $\alpha = (0, 0)$ we have

$$\sup_{(x,t) \in \mathbb{R} \times [0,\infty)} |\hat{u}(\varepsilon, x, t)| \leqslant \sup_{x \in \mathbb{R}} |\hat{u}_0(\varepsilon, x)|,$$

thus $\hat{u}$ satisfies (5). We prove by induction that $\partial_x^n \hat{u}$ satisfies (5) for each $n \in \mathbb{N}$. Denoting by

$$E(\varepsilon, x, t) = \exp\left(\frac{-x^2}{4\hat{v}(\varepsilon)\, t}\right) \quad \text{and} \quad F(\varepsilon, x) = \exp\left(-\frac{1}{2\hat{v}(\varepsilon)} \int_0^x \hat{u}_0(\varepsilon, \xi)\, d\xi\right),$$

we have

$$\hat{u}(\varepsilon, \cdot, t) = \frac{E(\varepsilon, \cdot, t) * (\hat{u}_0 F)(\varepsilon, \cdot)}{E(\varepsilon, \cdot, t) * F(\varepsilon, \cdot)}.$$

We will prove that

$$\partial_x^n \hat{u}(\varepsilon, \cdot, t) = \frac{\sum_{i=1}^{m(n)} \prod_{j=1}^{2^n} E(\varepsilon, \cdot, t) * (V_{ij} F)(\varepsilon, \cdot)}{[E(\varepsilon, \cdot, t) * F(\varepsilon, \cdot)]^{2^n}}, \tag{9}$$

where $m(n) \in \mathbb{N}$ and each V_{ij} is a sum of products of derivatives of $\hat{u}_0$ and powers of $1/\hat{v}(\varepsilon)$. In fact, for $n = 1$, we have

$$\begin{aligned}
\partial_x \hat{u}(\varepsilon, \cdot, t) &= \frac{1}{(E * F)^2} \\
&\quad \times \{[E * (\hat{u}_o' F + \hat{u}_o F')](E * F) - (E * F')(E * \hat{u}_o F)\} \\
&= \frac{1}{(E * F)^2} \left\{\left[E * \left(\hat{u}_o' - \frac{1}{2\hat{v}(\varepsilon)} \hat{u}_o^2\right) F\right] (E * F) \right. \\
&\quad \left. + \left(E * \frac{1}{2\hat{v}(\varepsilon)} \hat{u}_o F\right)(E * \hat{u}_o F)\right\}.
\end{aligned}$$

Assuming that (9) holds for n, we have

$$\begin{aligned}
\partial_x^{n+1} \hat{u}(\varepsilon, \cdot, t) &= \frac{1}{(E * F)^{2^{n+1}}} \left\{ \sum_{i=1}^{m(n)} \sum_{k=1}^{2^n} \prod_{j=1, j \neq k}^{2^n} E * (V_{ij} F) \cdot E * \left[(\partial_x V_{ik}) F \right.\right. \\
&\quad \left. + V_{ik}\left(-\frac{1}{2\hat{v}(\varepsilon)} \hat{u}_0 F\right)\right] (E * F)^{2^n} \\
&\quad \left. - \sum_{i=1}^{m(n)} \prod_{j=1}^{2^n} (E * V_{ij} F)\, 2^n (E * F)^{2^n - 1} \cdot E * \left(-\frac{1}{2\hat{v}(\varepsilon)} \hat{u}_o F\right)\right\} \\
&= \frac{1}{(E * F)^{2^{n+1}}} \sum_{i=1}^{m(n+1)} \prod_{j=1}^{2^{n+1}} (E * \tilde{V}_{ij} F),
\end{aligned}$$

where $m(n+1) \leqslant m(n)(2^{n+1}+1)$. Then (9) is true for all n and from it we obtain

$$\sup_{(x,t)\in\mathbb{R}\times(0,\infty)} |\partial_x^n \hat{u}(\varepsilon, x, t)| \leqslant \frac{1}{(E*F)^{2^n}} \sum_{i=1}^{m(n)} \sup_{i,j} \|V_{ij}\|_\infty^{2^n} (E*F)^{2^n}$$

$$= \sum_{i=1}^{m(n)} \sup_{i,j} \|V_{ij}\|_\infty^{2^n}$$

which has a bound c/ε^N, since $\hat{u}_o \in \mathscr{E}_{M,s,g}[\mathbb{R}]$ and $\hat{v}(\varepsilon)$ satisfies (8).

Concerning the t-derivatives and the mixed ones, it suffices to consider the differential equation

$$(\hat{u}_t + \hat{u}\hat{u}_x)(\varepsilon, x, t) = \hat{v}(\varepsilon)\, \hat{u}_{xx}(\varepsilon, x, t)$$

and by differentiation we obtain inequality (5) successively for $\hat{u}_t, \hat{u}_{tx}, \hat{u}_{txx}, \ldots, \hat{u}_{tt}, \hat{u}_{ttx}, \hat{u}_{ttxx}, \ldots$ and so on. ∎

In order to obtain our first uniqueness result we shall need the following Gronwall-type inequality.

2.2. LEMMA. *Let u be a nonnegative, continuous function on $[0, \infty)$ and assume that*

$$u(t) \leqslant a + b \int_0^t \frac{u(t_1)}{\sqrt{t-t_1}}\, dt_1$$

for some constants $a, b \geqslant 0$ and every $t \geqslant 0$. Then

$$u(t) \leqslant a(1 + 2b\sqrt{t}) \exp(\pi b^2 t).$$

Proof. Substitution of the equation into itself and evaluation of the integrals give

$$u(t) \leqslant a + b \int_0^t \frac{a}{\sqrt{t-t_1}}\, dt_1 + b^2 \int_0^t \int_0^{t_1} \frac{1}{\sqrt{t-t_1}} \frac{1}{\sqrt{t_1-t_2}}\, u(t_2)\, dt_2\, dt_1$$

$$= a(1 + 2b\sqrt{t}) + \pi b^2 \int_0^t u(t_2)\, dt_2.$$

Application of the usual Gronwall's inequality leads to the desired result. ∎

2.3. THEOREM. *Suppose that v is a real valued generalized constant with a representative $\hat{v}$ satisfying: there is $\eta > 0$ such that*

$$\hat{v}(\varepsilon) \log \frac{1}{\varepsilon} \geqslant 1 \tag{10}$$

for each $0<\varepsilon<\eta$. *Then for each* $T>0$ *there is at most one solution* $u\in\mathscr{G}_{s,g}(\mathbb{R}\times[0,T])$ *of* (1) *which is of bounded type.*

Proof. Let $u_1, u_2 \in \mathscr{G}_{s,g}(\mathbb{R}\times[0,T])$ be solutions of (1) with respective representatives $\hat{u}_1, \hat{u}_2$ of bounded type (7). Then there are $N\in\mathscr{N}_{s,g}(\mathbb{R}\times[0,T])$ and $n\in\mathscr{N}_{s,g}(\mathbb{R})$ such that

$$[(\hat{u}_1-\hat{u}_2)_t+\tfrac{1}{2}(\hat{u}_1^2-\hat{u}_2^2)_x+N](\varepsilon,x,t)=\hat{v}(\varepsilon)(\hat{u}_1-\hat{u}_2)_{xx}(\varepsilon,x,t)$$
$$(\hat{u}_1-\hat{u}_2)(\varepsilon,x,0)=n(\varepsilon,x).$$

By Duhamel's principle we have, for $t>0$,

$$\begin{aligned}(\hat{u}_1-\hat{u}_2)(\varepsilon,x,t)=&\int_{-\infty}^{+\infty}G(\varepsilon,x,t,x_1,0)\,n(\varepsilon,x_1)\,dx_1\\&-\int_0^t\int_{-\infty}^{+\infty}G(\varepsilon,x,t,x_1,t_1)\,N(\varepsilon,x_1,t_1)\,dx_1\,dt_1\\&+\int_0^t\int_{-\infty}^{+\infty}\frac{\partial G}{\partial x_1}(\varepsilon,x,t,x_1,t_1)\,\frac{1}{2}(\hat{u}_1^2-\hat{u}_2^2)(\varepsilon,x_1,t_1)\,dx_1\,dt_1,\end{aligned}$$

where

$$G(\varepsilon,x,t,x_1,t_1)=\frac{\exp\left[\dfrac{-(x-x_1)^2}{4(t-t_1)\,\hat{v}(\varepsilon)}\right]}{2\sqrt{\pi(t-t_1)\,\hat{v}(\varepsilon)}}$$

is the heat convolution kernel.

Now we observe that

$$\int_{-\infty}^{+\infty}G(\varepsilon,x,t,x_1,t_1)\,dx_1=1 \qquad\text{and}\qquad \int_{-\infty}^{+\infty}\left|\frac{\partial G}{\partial x_1}\right|dx_1=\frac{1}{\sqrt{\pi(t-t_1)\,\hat{v}(\varepsilon)}}.$$

Thus we obtain the estimate

$$\begin{aligned}\sup_{x\in\mathbb{R}}|(\hat{u}_1-\hat{u}_2)(\varepsilon,x,t)|\leqslant&\sup_{x_1\in\mathbb{R}}|n(\varepsilon,x_1)|+\int_0^t\sup_{x_1\in\mathbb{R}}|N(\varepsilon,x_1,t_1)|\,dt_1\\&+\int_0^t\frac{1}{\sqrt{\pi(t-t_1)\,\hat{v}(\varepsilon)}}\,\frac{1}{2}\sup_{x_1\in\mathbb{R}}|(\hat{u}_1-\hat{u}_2)(\varepsilon,x_1,t_1)|\,dt_1\\&\cdot\sup_{(x_1,t_1)\in\mathbb{R}\times[0,T]}|(\hat{u}_1+\hat{u}_2)(\varepsilon,x_1,t_1)|.\end{aligned}$$

Fixing $T>0$ we conclude by Lemma 2.2 that

$$\sup_{(x,t)\in\mathbb{R}\times[0,T]}|(\hat{u}_1-\hat{u}_2)(\varepsilon,x,t)|\leqslant a(1+2b\sqrt{T})\exp(\pi b^2T)$$

with

$$a = \sup_{x_1 \in \mathbb{R}} |n(\varepsilon, x_1)| + T \sup_{(x_1, t_1) \in \mathbb{R} \times [0, T]} |N(\varepsilon, x_1, t_1)|$$

and

$$b = \frac{1}{2\sqrt{\pi \hat{v}(\varepsilon)}} \sup_{(x_1, t_1) \in \mathbb{R} \times [0, T]} |(\hat{u}_1 + \hat{u}_2)(\varepsilon, x_1, t_1)|.$$

From our hypotheses on $\hat{v}(\varepsilon)$, n, N, and $\hat{u}_i$, $i = 1, 2$, we obtain inequality (6) for $\hat{u}_1 - \hat{u}_2$.

For the x-derivatives of $\hat{u}_1 - \hat{u}_2$ we have

$$\begin{aligned} \partial_x^m (\hat{u}_1 - \hat{u}_2) = & \int_{-\infty}^{+\infty} G \partial_{x_1}^m n \, dx_1 - \int_0^t \int_{-\infty}^{+\infty} G \partial_{x_1}^m N \, dx_1 \, dt_1 \\ & + \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \partial_{x_1} G \sum_{k=1}^{m} \binom{m}{k} \\ & \times [\partial_{x_1}^{m-k} (\hat{u}_1 - \hat{u}_2)] \, \partial_{x_1}^k (\hat{u}_1 + \hat{u}_2) \, dx_1 \, dt_1 \\ & + \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} \partial_{x_1} G \, [\partial_{x_1}^m (\hat{u}_1 - \hat{u}_2)] \, (\hat{u}_1 + \hat{u}_2) \, dx_1 \, dt_1. \end{aligned}$$

Assuming that, for $k < m$, $\partial_x^k (\hat{u}_1 - \hat{u}_2)$ has already been shown to satisfy (6), and using that $n \in \mathcal{N}_{s,g}(\mathbb{R})$, $N \in \mathcal{N}_{s,g}(\mathbb{R} \times [0, T])$ we obtain again from Lemma 2.2 that $\partial_x^m (\hat{u}_1 - \hat{u}_2)$ satisfies (6) as well. For the t and the mixed derivatives the argument is the same as in the proof of Theorem 2.1. ∎

We remark that the boundedness condition in Theorem 2.3 could be replaced by suitable growth requirements as $\varepsilon \to 0$, depending on the rate with which $\hat{v}(\varepsilon)$ approaches zero. For another uniqueness result we use a maximum principle which we state below:

2.4. Lemma. *Let*

$$\mathcal{L} u \equiv u_t - a_{ij}(x, t) \, u_{x_i x_j} + a_i(x, t) \, u_{x_i} + a(x, t) \, u = f(x, t),$$

where u is continuous at all points $(x, t) \in \mathbb{R}^n \times [0, T]$, has continuous derivatives u_t, u_x, and u_{xx}, satisfies the equation for $0 < t \leqslant T$, is bounded, the moduli of the coefficients a_{ij}, a_i do not exceed c and $a(x, t) \geqslant -a_o$, where c and a_o are nonnegative constants. Then the estimate

$$\sup_{x \in \mathbb{R}^n, 0 \leqslant t \leqslant T} |u(x, t)| \leqslant \Big(\sup_{x \in \mathbb{R}^n} |u(x, 0)| + T \sup_{x \in \mathbb{R}^n, 0 \leqslant t \leqslant T} |f(x, t)| \Big) \exp(a_o T)$$

is valid if $a_{ij}(x, t) \, \xi_i \xi_j \geqslant 0$ for all $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$.

Proof. See [11, Chap. I, Theorem 2.5].

2.5. THEOREM. *If ν is a positive generalized constant then for each $T>0$ there is at most one solution $u \in \mathscr{G}_{s,g}(\mathbb{R} \times [0, T])$ of* (1) *with $\partial_x u$ either of log-type or with a representative $\hat{u}$ satisfying: there is $\eta > 0$ such that $\partial_x \hat{u}(\varepsilon, x, t) \geqslant 0$ for all $(x, t) \in \mathbb{R} \times [0, T]$ and $0 < \varepsilon < \eta$.*

Proof. If u_1, u_2 are solutions of (1), with representatives $\hat{u}_1, \hat{u}_2$ both satisfying one of the hypotheses then there are $N \in \mathscr{N}_{s,g}(\mathbb{R} \times [0, T])$ and $n \in \mathscr{N}_{s,g}(\mathbb{R})$ such that

$$(\hat{u}_1 - \hat{u}_2)_t - \hat{\nu}(\hat{u}_1 - \hat{u}_2)_{xx} + \hat{u}_2(\hat{u}_1 - \hat{u}_2)_x + (\hat{u}_1)_x(\hat{u}_1 - \hat{u}_2) = N$$
$$(\hat{u}_1 - \hat{u}_2)(\varepsilon, x, 0) = n(\varepsilon, x).$$

From Lemma 2.4 the estimate

$$\sup_{x \in \mathbb{R}, 0 \leqslant t \leqslant T} |(\hat{u}_1 - \hat{u}_2)(\varepsilon, x, t)| \leqslant [\sup_{x \in \mathbb{R}} |n(\varepsilon, x)| + T \sup_{x \in \mathbb{R}, 0 \leqslant t \leqslant T} |N(\varepsilon, x, t)|] \exp(CT)$$

is valid, where $C = \max(0, \sup_{x,t} - (\hat{u}_1)_x)$. Thus $\hat{u}_1 - \hat{u}_2$ satisfies (6). For the x-derivatives, note that $\partial_x^m(\hat{u}_1 - \hat{u}_2)$ satisfies an equation of the form

$$\begin{aligned}[\partial_x^m(\hat{u}_1 - \hat{u}_2)]_t &- \hat{\nu}[\partial_x^m(\hat{u}_1 - \hat{u}_2)]_{xx} + \hat{u}_2[\partial_x^m(\hat{u}_1 - \hat{u}_2)]_x \\ &+ (\hat{u}_1 + m\hat{u}_2)_x \, \partial_x^m(\hat{u}_1 - \hat{u}_2) \\ &= f(\partial_x^m N, \hat{u}_1 - \hat{u}_2, ..., \partial_x^{m-1}(\hat{u}_1 - \hat{u}_2), \varepsilon, x, t).\end{aligned}$$

The estimate holds,

$$\max_{x \in \mathbb{R}, 0 \leqslant t \leqslant T} |\partial_x^m(\hat{u}_1 - \hat{u}_2)(\varepsilon, x, t)| \leqslant [\max_{x \in \mathbb{R}} |\partial_x^m n(\varepsilon, x)| + T \max_{x \in \mathbb{R}, 0 \leqslant t \leqslant T} |f|] \exp(C'T),$$

where $C' = \max(0, \sup_{x,t,i} - (m+1)(\hat{u}_i)_x)$. Since f is a sum of derivatives of N and products of the form $\partial_x^k(\hat{u}_1 - \hat{u}_2)\, \partial_x^l(\hat{u}_1 + j\hat{u}_2)$, $0 \leqslant k < m$, $0 \leqslant l \leqslant m+1$, it follows that $\partial_x^m(\hat{u}_1 - \hat{u}_2)$ satisfies (6). For the t-derivatives and the mixed ones the argument is the same as in the proof of Theorem 2.1. ∎

2.6. THEOREM. *The map $f: u_o \in \mathscr{G}_{s,g}(\mathbb{R}) \to u \in \mathscr{G}_{s,g}(\mathbb{R} \times [0, T])$, where u is the unique solution of* (1) *(with ν satisfying* (8)*) satisfying one of the two requisites of Theorem* 2.5, *is continuous, when both spaces $\mathscr{G}_{s,g}$ are endowed with the topology defined in Definition* 1.5.

Proof. Let $u_1, u_2 \in \mathscr{G}_{s,g}(\mathbb{R} \times [0, T])$ be the solutions to (1) with respective initial data $u_{o,1}, u_{o,2} \in \mathscr{G}_{s,g}(\mathbb{R})$. Then

$$(\hat{u}_1 - \hat{u}_2)_t - \hat{v}(\hat{u}_1 - \hat{u}_2)_{xx} + \hat{u}_2(\hat{u}_1 - \hat{u}_2)_x + (\hat{u}_1)_x(\hat{u}_1 - \hat{u}_2) = N$$

$$(\hat{u}_1 - \hat{u}_2)(\varepsilon, x, 0) = (\hat{u}_{o,1} - \hat{u}_{o,2} + n)(\varepsilon, x),$$

with $n \in \mathscr{N}_{s,g}(\mathbb{R})$, $N \in \mathscr{N}_{s,g}(\mathbb{R} \times [0, T])$. By Lemma 2.4,

$$\begin{aligned}&\|\hat{u}_1 - \hat{u}_2\|_{L^\infty(\mathbb{R} \times [0,T])}\\ &\quad \leqslant \{\|\hat{u}_{o,1} - \hat{u}_{o,2} + n\|_{L^\infty(\mathbb{R})} + T\,\|N\|_{L^\infty(\mathbb{R} \times [0,T])}\} \exp(CT)\end{aligned}$$

with C as in the proof of Theorem 2.5. By the hypotheses on $\hat{u}_1, \hat{u}_2$, we have that $\exp(CT) \leqslant C_1 \varepsilon^{-M}$ for some $C_1 > 0$, $M \in \mathbb{N}$ and all $\varepsilon > 0$ small enough.

Given a 0-neighborhood $V(0, q)$ in $\mathscr{G}_{s,g}(\mathbb{R} \times [0, T])$, we have that $u_1 - u_2 \in V(0, q)$ provided $u_{o,1} - u_{o,2}$ belongs to the 0-neighborhood $V(0, q + N + 1)$ in $\mathscr{G}_{s,g}(\mathbb{R})$.

For the first x-derivative, we have

$$\begin{aligned}((\hat{u}_1 - \hat{u}_2)_x)_t - \hat{v}((\hat{u}_1 - \hat{u}_2)_x)_{xx} + \hat{u}_2((\hat{u}_1 - \hat{u}_2)_x)_x&\\ + (\hat{u}_1 + \hat{u}_2)_x\,(\hat{u}_1 - \hat{u}_2)_x = N' - (\hat{u}_1)_{xx}\,(\hat{u}_1 - \hat{u}_2)&\\ (\hat{u}_1 - \hat{u}_2)|_{t=0} = \hat{u}'_{o,1} - \hat{u}'_{o,2} + n'.&\end{aligned}$$

Then

$$\begin{aligned}\|(\hat{u}_1 - \hat{u}_2)_x\|_{L^\infty(\mathbb{R} \times [0,T])} \leqslant \exp(C'T)\,\{&\|\hat{u}'_{o,1} - \hat{u}'_{o,2} + n'\|_\infty\\ &+ T\|(\hat{u}_1)_{xx}\|_\infty\,\|\hat{u}_1 - \hat{u}_2\|_\infty + T\|N'\|_\infty\end{aligned}$$

with C' as in the case $m = 1$ of the proof of Theorem 2.5. Then we can achieve that $(u_1 - u_2)_x \in V(1, q)$, if we take $u_{o,1} - u_{o,2} \in V(1, q')$ with q' so large that $\|u_1 - u_2\|_\infty$ is so small as to compensate both $\exp(C'T)$ and $\|(\hat{u}_1)_{xx}\|_\infty$. Now go on by induction. ∎

2.7. *Remark.* The topology defined by the $Q(p, \mu)$ in [1] would not work even for the 0th derivative because of the growth of $\exp(CT)$ in $1/\varepsilon$. It would work, though, on generalized functions all whose derivatives are of bounded type.

3. Relation to Classical Solutions, Delta Waves

In [12], Lax introduced the following pseudo-norm, defined on locally integrable functions f on $\mathbb{R}$, but possibly infinite:

$$|f|_* = \sup_{z \in \mathbb{R}} \left| \int_0^z f(x)\, dx \right|. \tag{11}$$

Employing some ideas of Lax [12], we obtain a general result on continuous dependence:

3.1. Proposition. *Let $u_{o,i} \in L^\infty(\mathbb{R})$, $v_i \geqslant 0$ and u_i be the classical solution to*

$$u_t + uu_x = v_i u_{xx}$$
$$u(x, 0) = u_{o,i}(x)$$

for $i = 1, 2$ and assume that $|u_{o,1} - u_{o,2}|_*$ is finite. Then, for every $T > 0$,

$$\sup_{0 \leqslant t \leqslant T} |u_1(\cdot, t) - u_2(\cdot, t)|_* \leqslant 2\, |u_{o,1} - u_{o,2}|_* + 2T\, |v_1 - v_2| \cdot \|(u_2)_x\|_{L^\infty(\mathbb{R} \times (0, T))}. \tag{12}$$

Proof. Let $U_i(x, t) = \int_0^x u_i(\xi, t)\, d\xi + h_i(t)$, where

$$h_i'(t) = v_i (u_i)_x (0, t) - \tfrac{1}{2} u_i^2 (0, t), \qquad h_i(0) = 0,$$

$i = 1, 2$. Then U_i satisfies

$$(U_i)_t + \tfrac{1}{2}(U_i)_x^2 = v_i (U_i)_{xx}, \quad U_i(x, 0) = \int_0^x u_{o,i}(\xi)\, d\xi.$$

Thus, letting $W = U_1 - U_2$,

$$\begin{aligned} W_t + \tfrac{1}{2}(u_1 + u_2)\, W_x &= v_1 W_{xx} + (v_1 - v_2)(u_2)_x \\ W(x, 0) &= \int_0^x [u_{0,1}(\xi) - u_{o,2}(\xi)]\, d\xi. \end{aligned} \tag{13}$$

The function W is smooth for $t > 0$ and continuous up to $t = 0$ (see [10, Theorem 1]) and grows linearly as $|x| \to \infty$, while $W(x, 0)$ is bounded by assumption. Under this growth restriction the assertion of Lemma 2.4 is still valid (as can be seen immediately from the proof in [11]) and we conclude that

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}} |W(x, t)| \leqslant \sup_{x \in \mathbb{R}} |W(x, 0)| + T\, |v_1 - v_2| \sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}} |(u_2)_x(x, t)|.$$

Since $|u_1(\cdot, t) - u_2(\cdot, t)|_* \leqslant \sup_{x \in \mathbb{R}} |W(x, t)| + |W(0, t)|$ the estimate (12) follows. ∎

3.2. LEMMA. *If $|f_\varepsilon|_* \to 0$ as $\varepsilon \to 0$ then $f_\varepsilon \to 0$ in $\mathscr{D}'(\mathbb{R})$. The converse is true if $(f_\varepsilon)_{\varepsilon > 0}$ is a bounded subset of $L^\infty(\mathbb{R})$ and the supports of the functions f_ε are contained in a common compact set.*

Proof. A straightforward integration by parts shows the first part. Let $M > 0$ be such that $\|f_\varepsilon\|_\infty \leqslant M$. If $|f_\varepsilon|_* \not\to 0$ as $\varepsilon \to 0$ we can find $\delta > 0$ and a sequence $\varepsilon_k \to 0$ such that $\sup_{x \in \mathbb{R}} |\int_0^x f_{\varepsilon_k}(\xi)\, d\xi| \geqslant \delta$ for all k. Since $x \mapsto \int_0^x f_\varepsilon(\xi)\, d\xi$ is continuous we can find x_k such that $|\int_0^{x_k} f_{\varepsilon_k}(\xi)\, d\xi| \geqslant \delta$. If K is a compact subset that contains $\operatorname{supp} f_\varepsilon$ for all $\varepsilon > 0$, then $\int_0^x f_{\varepsilon_k}(\xi)\, d\xi$ is constant outside K and so the x_k may be chosen to belong to K. Hence they have an accumulation point z. But then (observing that $\|f_\varepsilon\|_\infty \leqslant M$), we have $|\int_0^z f_{\varepsilon_k}(\xi)\, d\xi| \geqslant \delta/2$ for infinitely many k. If we take $\varphi \in \mathscr{D}$, $\operatorname{supp} \varphi \subset [0, z]$, $\varphi \equiv 1$ on $[\alpha, z - \alpha]$ for small α, we shall have that

$$\left| \int_{-\infty}^{\infty} f_{\varepsilon_k}(\xi)\, \varphi(\xi)\, d\xi \right| \geqslant \frac{\delta}{3}$$

and so $f_\varepsilon \not\to 0$ in $\mathscr{D}'$. ∎

As a first application, we can deduce a coherence result relating the generalized solution in $\mathscr{G}_{s,g}(\mathbb{R} \times [0, \infty))$ to the classical solution, when the latter exists.

3.3. COROLLARY. *Let $u_0 \in L^\infty(\mathbb{R})$ and let u be the classical entropy solution to (3). Further let v be as in Theorem 2.1 and $v \approx 0$. Finally, let $v \in \mathscr{G}_{s,g}(\mathbb{R} \times [0, \infty))$ be the solution to*

$$v_t + v v_x = \nu v_{xx}$$
$$v|_{t=0} = \operatorname{class}[\iota_\rho(u_o)]$$

given by Theorem 2.1, where ι_ρ is as described in Remark 1.4. Then $v \approx u$.

For the proof we need a lemma:

3.4. LEMMA. *Let $w \in L^\infty(\mathbb{R})$ and $\rho \in \mathscr{S}(\mathbb{R})$ as in Remark 1.4. Then $|w - w * \rho_\varepsilon|_* \to 0$ as $\varepsilon \to 0$.*

Proof. In fact,

$$\left|\int_0^z (w - w * \rho_\varepsilon)(x)\,dx\right| = \left|\int_{-\infty}^{+\infty} \rho(y) \int_0^z [w(x) - w(x-\varepsilon y)]\,dx\,dy\right|$$
$$= \left|\int_{-\infty}^{+\infty} \rho(y)\left[\int_0^z w(x)\,dx - \int_{-\varepsilon y}^{z-\varepsilon y} w(x)\,dx\right] dy\right|$$
$$\leqslant 2\varepsilon\,\|w\|_\infty \int_{-\infty}^{+\infty} |y\rho(y)|\,dy \to 0$$

as $\varepsilon \to 0$ uniformly for $z \in \mathbb{R}$. ∎

Proof of Corollary 3.3. Let $\hat{v}$ be a representative of v satisfying

$$\hat{v}_t(\varepsilon, x, t) + \hat{v}(\varepsilon, x, t)\,\hat{v}_x(\varepsilon, x, t) = \hat{v}(\varepsilon)\,\hat{v}_{xx}(\varepsilon, x, t)$$
$$\hat{v}(\varepsilon, x, 0) = u_0 * \rho_\varepsilon(x).$$

From Lemma 3.4, $|u_o - u_o * \rho_\varepsilon|_* \to 0$ as $\varepsilon \to 0$. Applying Proposition 3.1 with $v_1 = v_2 = \hat{v}(\varepsilon)$, where $\hat{v}$ is a representative of v, $u_{o,1} = u_o$, $u_{o,2} = u_o * \rho_\varepsilon$ for fixed ε, we obtain for the classical solutions u_ε and $\hat{v}(\varepsilon, \cdot)$,

$$\sup_{0 \leqslant t \leqslant T} |u_\varepsilon(\cdot, t) - \hat{v}(\varepsilon, \cdot, t)|_* \leqslant 2|u_0 - u_0 * \rho_\varepsilon|_*.$$

Since u_ε converges to the entropy solution u in $\mathscr{D}'(\mathbb{R} \times (0, \infty))$ (see [14, p. 142]), the assertion follows. ∎

3.5. *Remarks.* (1) In a similar way one can prove that if v is a fixed positive real number, v is as in Corollary 3.3 and u_v is the classical solution to (1), then $v \approx u_v$.

(2) If $\lim_{x \to \pm\infty} u_0(x)$ exists, then the classical solutions to $u_t + uu_x = vu_{xx}$ with initial data u_o converge to $\frac{1}{2}(u_o(-\infty) + u_o(\infty))$ as $v \to \infty$. Hence, if v is a constant generalized function associated with infinity, $v \approx \infty$, and v is the generalized solution of Corollary 3.3, then $v \approx \frac{1}{2}(u_o(-\infty) + u_o(\infty))$. In addition, for $u_o \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, one can prove that $v \approx 0$.

3.6. THEOREM. *Let $u_o \in \mathscr{G}_{s,g}(\mathbb{R})$ be a Dirac generalized function, i.e., it has a representative $\hat{u}_o$ satisfying*:

(1) $\hat{u}_0(\varepsilon, x) \geqslant 0$ *for all* $x \in \mathbb{R}$ *and* $\varepsilon > 0$;

(2) $\hat{u}_0(\varepsilon, x) = 0$ *if* $|x| \geqslant a(\varepsilon)$ *for some* $a(\varepsilon) \to 0$ *as* $\varepsilon \to 0$;

(3) $\int_{-\infty}^{+\infty} \hat{u}_o(\varepsilon, x)\,dx = 1$ *for all* $\varepsilon > 0$.

In particular, $u_0 \approx \delta$. *Let* $u_{\nu,\varepsilon}$ *be the classical solution to*

$$\begin{aligned} u_t + uu_x &= \nu u_{xx} \\ u(x, 0) &= \hat{u}_o(\varepsilon, x) \end{aligned} \tag{14}$$

with $\nu > 0$ *and* $\varepsilon > 0$. *Then*

$$\lim_{(\nu,\varepsilon) \to (0,0)} u_{\nu,\varepsilon} = \chi_{\{0 < x < \sqrt{2t}\}}(x, t) \cdot \frac{x}{t} \qquad in \quad \mathscr{D}'(\mathbb{R} \times (0, \infty)).$$

In particular, whenever ν *is a positive generalized constant such that* $\nu \approx 0$, *the solution* $u_\nu \in \mathscr{G}_{s,g}(\mathbb{R} \times (0, \infty))$ *to* (1) *is associated with* $\chi_{\{0 < x < \sqrt{2t}\}}(x, t) \cdot (x/t)$.

Proof. Let, for $\nu > 0$ and $\varepsilon > 0$,

$$H_{\nu,\varepsilon}(x, t) = \frac{1}{2\sqrt{\nu t}} \int_{-\infty}^{+\infty} \exp\left(\frac{-y^2}{4\nu t}\right) \exp\left(-\frac{1}{2\nu} \int_{-\infty}^{x-y} \hat{u}_o(\varepsilon, \xi)\, d\xi\right) dy.$$

Then the classical solution to (14) is given by

$$u_{\nu,\varepsilon}(x, t) = -2\nu \partial_x \log\left(\frac{1}{\sqrt{\pi}} H_{\nu,\varepsilon}\right)(x, t) = -2\nu \partial_x (\log H_{\nu,\varepsilon})(x, t).$$

Given $\psi \in \mathscr{D}(\mathbb{R} \times (0, \infty))$, we have

$$\int_0^\infty \int_{-\infty}^{+\infty} u_{\nu,\varepsilon}(x, t)\, \psi(x, t)\, dx\, dt = 2\nu \int_0^{+\infty} \int_{-\infty}^{+\infty} \log H_{\nu,\varepsilon}(x, t)\, \partial_x \psi(x, t)\, dx\, dt.$$

Let us find uniform bounds for $|2\nu \log H_{\nu,\varepsilon}(x, t)|$ with (x, t) in compact sets of $\mathbb{R} \times (0, \infty)$: observing that

$$\exp\left(-\frac{1}{2\nu} \int_{-\infty}^{x-y} \hat{u}_o(\varepsilon, \xi)\, d\xi\right) = \begin{cases} 1 & \text{if} \quad y \geqslant x + a(\varepsilon) \\ \exp(-1/2\nu) & \text{if} \quad y \leqslant x - a(\varepsilon) \end{cases}$$

we have that

$$\begin{aligned} H_{\nu,\varepsilon}(x, t) = \frac{1}{2\sqrt{\nu t}} \bigg\{ &\int_{x+a(\varepsilon)}^{\infty} \exp\left(\frac{-y^2}{4\nu t}\right) dy \\ &+ \exp\left(\frac{-1}{2\nu}\right) \int_{-\infty}^{x-a(\varepsilon)} \exp\left(\frac{-y^2}{4\nu t}\right) dy \\ &+ \int_{x-a(\varepsilon)}^{x+a(\varepsilon)} \exp\left(\frac{-y^2}{4\nu t}\right) \exp\left(-\frac{1}{2\nu} \int_{-\infty}^{x-y} \hat{u}_o(\varepsilon, \xi)\, d\xi\right) dy \bigg\}. \end{aligned}$$

Thus $0 \leqslant H_{\nu,\varepsilon}(x, t) \leqslant 3\sqrt{\pi}$. We need an estimate away from zero: the first term of $H_{\nu,\varepsilon}$ is $\geqslant \sqrt{\pi}/2$ if $x + a(\varepsilon) \leqslant 0$ and $\geqslant \frac{1}{4}\exp(-2(x+a(\varepsilon))^2/4\nu t)$ if $x + a(\varepsilon) \geqslant 0$ (using the estimate $\exp(-y^2) \geqslant y \exp(-2y^2)$ for $y \geqslant 0$). Then

$$|2\nu \log H_{\nu,\varepsilon}| \leqslant \max\left\{2\nu \log(3\sqrt{\pi}),\ 2\nu\left|\log\frac{\sqrt{\pi}}{2}\right|,\ \left|2\nu\log\frac{1}{4} - \frac{(x+a(\varepsilon))^2}{t}\right|\right\}.$$

By Lebesgue's dominated convergence theorem, we may now calculate the limit pointwise a.e.

For $x < 0$ and $\varepsilon > 0$ small enough such that $x + a(\varepsilon) \leqslant 0$ we see that

$$2\nu\,|\log H_{\nu,\varepsilon}(x, t)| \leqslant 2\nu \log(3\sqrt{\pi}) \to 0 \qquad \text{as} \quad \nu \to 0. \tag{15}$$

Thus let $x > 0$. We consider two cases:

(a) $0 < x < \sqrt{2t}$. Then, for $\varepsilon < \varepsilon(\delta)$, $\delta > 0$ given, we have

$$\int_{(x+\delta)/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\,dy \leqslant H_{\nu,\varepsilon}(x, t) \leqslant 2\int_{(x-\delta)/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\,dy, \tag{16}$$

observing that

$$\exp\left(-\frac{1}{2\nu}\right)\int_{-\infty}^{(x-a(\varepsilon))/2\sqrt{\nu t}} \exp(-y^2)\,dy \leqslant \int_{(x-\delta)/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\,dy$$

for small ν and $(x-\delta)^2 < 2t$, since then

$$\exp\left(\frac{1}{2\nu}\right)\int_{(x-\delta)/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\,dy \to +\infty \qquad \text{as} \quad \nu \to 0.$$

Thus, for ε and ν small enough, we have from (16) that

$$\begin{aligned}2\nu \log \int_{(x+\delta)/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\,dy &\leqslant 2\nu \log H_{\nu,\varepsilon}(x, t)\\ &\leqslant 2\nu\left(\log\int_{(x-\delta)/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\,dy + \log 2\right).\end{aligned}$$

Applying l'Hospital's rule twice, the formula

$$\lim_{\nu\to 0} 2\nu \log \int_{\xi/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\,dy = -\frac{\xi^2}{2t}$$

is seen to be valid if $\xi > 0$. We conclude that

$$-\frac{(x+\delta)^2}{2t} \leqslant \lim_{(\nu,\varepsilon)\to(0,0)} 2\nu \log H_{\nu,\varepsilon}(x, t) \leqslant -\frac{(x-\delta)^2}{2t}$$

for every $\delta > 0$, i.e.,

$$\lim_{(\nu,\varepsilon)\to(0,0)} 2\nu \log H_{\nu,\varepsilon}(x, t) = -\frac{x^2}{2t}. \tag{17}$$

(b) $x > \sqrt{2t}$. Then, for $\varepsilon < \varepsilon(\delta)$,

$$\begin{aligned}
&\int_{(x+\delta)/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\, dy + \exp\left(-\frac{1}{2\nu}\right) \int_{-\infty}^{(x-\delta)/2\sqrt{\nu t}} \exp(-y^2)\, dy \\
&\quad \leqslant H_{\nu,\varepsilon}(x, t) \\
&\quad \leqslant \int_{(x-\delta)/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\, dy + \exp\left(-\frac{1}{2\nu}\right) \int_{-\infty}^{x/2\sqrt{\nu t}} \exp(-y^2)\, dy.
\end{aligned}$$

But

$$\begin{aligned}
&2\nu \log\left(\int_{\xi/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\, dy + \exp\left(-\frac{1}{2\nu}\right) \int_{-\infty}^{\eta/2\sqrt{\nu t}} \exp(-y^2) dy\right) \\
&\quad = 2\nu \log\left(\exp\left(\frac{1}{2\nu}\right) \int_{\xi/2\sqrt{\nu t}}^{\infty} \exp(-y^2)\, dy + \int_{-\infty}^{\eta/2\sqrt{\nu t}} \exp(-y^2)\, dy\right) - 1.
\end{aligned} \tag{18}$$

The second integral inside the logarithm tends to $\sqrt{\pi}$ as $\nu \to 0$, if $\eta > 0$, while the first tends to zero if $\xi^2 > 2t$, again by l'Hospital's rule.

Thus (18) tends to -1 as $\nu \to 0$, and it follows that

$$\lim_{(\nu,\varepsilon)\to(0,0)} 2\nu \log H_{\nu,\varepsilon}(x, t) = -1. \tag{19}$$

Collecting all cases, (15, 17, 19), we finally arive at

$$\begin{aligned}
&\lim_{(\nu,\varepsilon)\to(0,0)} \int_0^\infty \int_{-\infty}^\infty u_{\nu,\varepsilon}(x, t)\, \psi(x, t)\, dx\, dt \\
&\quad = -\int_0^\infty \int_{-\infty}^\infty \left\{\begin{matrix} 0, & x \leqslant 0 \\ x^2/2t, & 0 < x < \sqrt{2t} \\ 1, & x > \sqrt{2t} \end{matrix}\right\} \partial_x \psi(x, t)\, dx\, dt \\
&\quad = \int_0^\infty \int_{-\infty}^\infty \frac{x}{t} \chi_{\{0 < x < \sqrt{2t}\}}(x, t)\, \psi(x, t)\, dx\, dt. \quad \blacksquare
\end{aligned}$$

We remark here that in the absence of viscosity, i.e., for Eq. (3), delta waves have been calculated by Gramchev [9]; see also the discussion in [13].

4. The Zero Viscosity Limits

As a second application of Theorem 3.1, we investigate the question of uniqueness of generalized solutions (in the sense of association) to Burgers' equation with zero viscosity. That is, we ask under what circumstances the solutions to (4) are unique, where $u_o \in \mathcal{G}_{s,g}(\mathbb{R})$ and $u \in \mathcal{G}_{s,g}(\mathbb{R} \times [0, T])$. Assume first that u has a representative $\hat{u}$ which is bounded uniformly in ε and so that $\hat{u}(\varepsilon, \cdot) \to w \in L^\infty$ a.e. as $\varepsilon \to 0$. Then $u_t + uu_x \approx 0$ if, and only if, w is a weak solution (in the classical sense) to the conservation law $w_t + (w^2/2)_x = 0$. Without further conditions these weak solutions are not unique, consequently neither are the solutions to (4).

An explicit example of nonuniqueness in the even stronger problem (2) with $u_o \in \mathcal{G}_{s,g}(\mathbb{R})$ is given next.

4.1. Example. Let $u_o \in \mathcal{G}_{s,g}(\mathbb{R})$ have a representative $\hat{u}_o(\varepsilon, \cdot)$ which is bounded uniformly in ε and converges to the Heaviside function a.e. as $\varepsilon \to 0$. Set

$$\hat{u}_1(\varepsilon, x, t) = \hat{u}_o(\varepsilon, x - t/2)$$

$$\hat{u}_2(\varepsilon, x, t) = -\hat{u}_o(\varepsilon, x + t/2) + 2\hat{u}_o(\varepsilon, x).$$

Letting $\psi \in \mathcal{D}(\mathbb{R} \times (0, \infty))$, we have for $i = 1, 2$,

$$\int_0^\infty \int_{-\infty}^\infty \left[\hat{u}_i(\varepsilon, x, t)\, \psi_t(x, t) + \frac{1}{2} \hat{u}_i^2(\varepsilon, x, t)\, \psi_x(x, t) \right] dx\, dt \to 0 \qquad \text{as} \quad \varepsilon \to 0$$

since the $\hat{u}_i(\varepsilon, \cdot, \cdot)$ both converge to classical weak solutions of the inviscid Burgers' equation. Thus the classes of $\hat{u}_1$ and $\hat{u}_2$ in $\mathcal{G}_{s,g}(\mathbb{R} \times [0, \infty))$ both are solutions of (2) with the same initial data.

4.2. Definition. A generalized function $u \in \mathcal{G}_{s,g}(\mathbb{R} \times [0, \infty))$ is called a *zero viscosity limit* if it satisfies the equation $u_t + uu_x = \nu u_{xx}$ for some positive generalized constant $\nu \approx 0$, and if in addition $\nu u_{xx} \approx 0$.

Clearly, if u is a zero viscosity limit then $u_t + uu_x \approx 0$. The existence of zero viscosity limits is guaranteed by the following proposition.

4.3. Proposition. *Under the hypotheses of Theorem* 2.1, *assume that* u_o *is of bounded type and that* $\nu \approx 0$. *Then, the solution* u *constructed there is a zero viscosity limit, and in particular, solves problems* (2) *and* (4).

Proof. The proof of Theorem 2.1 shows that u is of bounded type. Thus $\nu u \approx 0$, and so $\nu u_{xx} \approx 0$ as well. This proves that u is a zero viscosity limit. The other assertions follow trivially. ∎

We note that if u_o is of the form $\iota_\rho(v_o)$ for some $v_o \in L^\infty(\mathbb{R})$, then it is of bounded type. We ask whether solutions to (4) are unique (up to association) in the class of zero viscosity limits. Contrary to the classical setting, this turns out not to be true. In fact, we shall even show that solutions to (1) are not unique, when equality is understood in the sense of the more general space $\mathscr{G}_s(\mathbb{R} \times [0, \infty))$.

4.4. Example. Let v be a generalized constant having a representative $\hat{v}(\varepsilon)$ such that $\varepsilon^N \leqslant \hat{v}(\varepsilon) \leqslant N$ for some $N \in \mathbb{N}$. Then there exists an element $u \in \mathscr{G}_s(\mathbb{R} \times [0, \infty))$ with a representative $\hat{u}$ belonging to $\mathscr{E}_{M,s,g}[\mathbb{R} \times [0, \infty)]$ such that

$$u_t + uu_x = vu_{xx} \quad \text{in} \quad \mathscr{G}_s(\mathbb{R} \times [0, \infty))$$

$$u|_{t=0} = 0 \quad \text{in} \quad \mathscr{G}_s(\mathbb{R}),$$

but $u \neq 0$. In fact, u does not even admit an associated distribution for positive time. The starting point is the well-known family of solitons $(c > 0, x_o \in \mathbb{R})$,

$$(x, t) \mapsto c - c \tanh\left(\frac{c}{2v}(x - x_0 - ct)\right). \tag{20}$$

Define $\hat{u} \in \mathscr{E}_s[\mathbb{R} \times [0, \infty)]$ by

$$\hat{u}(\varepsilon, x, t) = \frac{1}{\varepsilon} - \frac{1}{\varepsilon} \tanh\left(\frac{1}{2\varepsilon\hat{v}(\varepsilon)}\left(x + \frac{1}{\varepsilon} - \frac{1}{\varepsilon} t\right)\right).$$

We claim that $\hat{u} \in \mathscr{E}_{M,s}[\mathbb{R} \times [0, \infty)]$. Indeed, any derivative of $\tanh y$ is a polynomial in $\tanh y$. It follows that any derivative of $\hat{u}(\varepsilon, \cdot)$ is bounded by some negative power of $\varepsilon\hat{v}(\varepsilon)$. Thus $\hat{u}$ belongs even to $\mathscr{E}_{M,s,g}[\mathbb{R} \times [0, \infty)]$. Next we show that $\hat{u}(\cdot, 0) \in \mathscr{N}_s(\mathbb{R})$. If x varies in a compact subset of $\mathbb{R}$, then eventually $x \geqslant 2 - 1/\varepsilon$. But then

$$\hat{u}(\varepsilon, x, 0) \leqslant \frac{1}{\varepsilon}\left(1 - \tanh\frac{1}{\varepsilon\hat{v}(\varepsilon)}\right) \leqslant \frac{2}{\varepsilon} \exp\left(-\frac{2}{\varepsilon\hat{v}(\varepsilon)}\right)$$

and this tends to zero faster than any power of ε. A similar argument applies to the derivatives (the fact that every derivative of $\tanh y$ has $(\cosh y)^{-2}$ as a factor yields the exponential decay of $\partial_x^j \hat{u}(\varepsilon, x, 0)$ as $\varepsilon \to 0$, uniformly on x in compact sets).

This shows that u, the class of $\hat{u}$ in $\mathscr{G}_s(\mathbb{R} \times [0, \infty))$ solves (1) with $u_o = 0$ in $\mathscr{G}_s(\mathbb{R})$. However, $u \neq 0$ in $\mathscr{G}_s(\mathbb{R} \times [0, \infty))$. In fact, it is seen immediately that $\hat{u}(\varepsilon, x, t) \geqslant 1/\varepsilon$ uniformly on $x \leqslant 0$, $t \geqslant 1$. Thus u does not admit an associated distribution.

4.5. *Remarks.* (a) Observe the subtlety that $u|_{t=0}=0$ in $\mathscr{G}_s(\mathbb{R})$, but $u|_{t=0}\neq 0$ in $\mathscr{G}_{s,g}(\mathbb{R})$, so Example 4.4 does not contradict our uniqueness results of Section 2.

(b) Taking $\hat{v}(\varepsilon)=o(\varepsilon)$ in Example 4.4 we obtain an element $u\in\mathscr{G}_{s,g}(\mathbb{R}\times[0,\infty))$ which is a zero viscosity limit and satisfies $u|_{t=0}\approx 0$. Thus u solves (4) with $u_o=0$, but $u\not\approx 0$. Hence two zero viscosity limit solutions to (4) need not be associated with each other.

(c) A similar example can be constructed starting from certain classical shock wave solutions to problem (3). Indeed, $u(x,t)=2c\Theta(x_o+ct-x)$, where Θ is the Heaviside function, is a classical weak solution to (3) for every $c>0$ (actually obtained from (20) by letting $v\to 0$). Taking $c=1/\varepsilon$, $x_o=-1/\varepsilon$ and smoothing Θ yields another example of an element $u\in\mathscr{G}_{s,g}(\mathbb{R}\times[0,\infty))$ which solves (4) with $u_o=0$, but does not admit an associated distribution.

Finally we shall prove uniqueness to problem (4), in particular, in the class of zero viscosity limits with compactly supported initial data satisfying certain boundedness conditions. It will be even allowed that the zero viscosity limits arise from different viscosity coefficients, provided they are not too far apart.

4.6. THEOREM. *For $i=1,2$, let $u_{o,i}\in\mathscr{G}_{s,g}(\mathbb{R})$ have compact support. Assume that $u_{o,i}$ and $u'_{o,i}$ are of bounded type. Let v_i be positive generalized constants satisfying (8) and $v_1/v_2\approx 1$. Let $u_i\in\mathscr{G}_{s,g}(\mathbb{R}\times[0,\infty))$ be the solutions to (1) with $v=v_i$ and $u_o=u_{o,i}$, given by Theorem 2.1. Then, if $u_{o,1}\approx u_{o,2}$, we have $u_1\approx u_2$.*

Proof. Observe that if $u_{o,i}$ has compact support then it also has a representative $\hat{u}_{o,i}$ so that the supports of all $\hat{u}_{o,i}(\varepsilon,\cdot)$ are contained in a common compact set. By Lemma 3.2, we infer from $u_{o,1}\approx u_{o,2}$ and the boundedness assumption on $\hat{u}_{o,i}$ that

$$|\hat{u}_{o,1}(\varepsilon,\cdot)-\hat{u}_{o,2}(\varepsilon,\cdot)|_*\to 0 \qquad \text{as} \quad \varepsilon\to 0.$$

In addition, it follows from the proof of Theorem 2.1 and the boundedness assumption on $\hat{u}'_{o,2}$ that the L^∞-norm of $\partial_x\hat{u}_2(\varepsilon,\cdot,\cdot)$ is bounded by a multiple of $\hat{v}_2(\varepsilon)^{-1}$. Thus Proposition 3.1 and the hypothesis $v_1/v_2\approx 1$ imply the result. ∎

4.7. *Remark.* If $u_{o,1}$ actually equals $u_{o,2}$, then the hypothesis of compact support can be dropped, and the conclusion of Theorem 4.6 remains valid. This way we obtain a result asserting uniqueness in the sense of association for zero viscosity limit solutions to problem (2), as indicated in the Introduction.

As a final application of Proposition 3.1, we give an example of a solution to (3) in $\mathscr{G}_s(\mathbb{R}\times[0,\infty))$ which is not C^∞.

4.8. EXAMPLE. Let $\hat{u}_o(\varepsilon, x)=\Theta * \varphi_\varepsilon(x)$, where Θ is the Heaviside function and $\varphi\in\mathscr{D}(\mathbb{R})$, $\varphi\geqslant 0$ and supp $\varphi\subset[-1,1]$. Since $\hat{u}'_o(\varepsilon, x)=\varphi_\varepsilon(x)\geqslant 0$ for all x, there is a global classical solution $\hat{u}(\varepsilon,\cdot)$ to (3) given implicitly by

$$\hat{u}(\varepsilon, x, t)=\hat{u}_o(\varepsilon, x-\hat{u}(\varepsilon, x, t)\, t)$$

with

$$\hat{u}_x(\varepsilon, x, t)=\frac{\varphi_\varepsilon(x)}{1+t\varphi_\varepsilon(x)}, \qquad \hat{u}_t(\varepsilon, x, t)=\frac{-\hat{u}(\varepsilon, x, t)\,\varphi_\varepsilon(x)}{1+t\varphi_\varepsilon(x)}.$$

Then $|\hat{u}(\varepsilon, x, t)|\leqslant 1$ for all (ε, x, t), and all derivatives of φ_ε are uniformly bounded by some power of $1/\varepsilon$. Thus it follows that $\hat{u}\in\mathscr{E}_{M,s,g}[\mathbb{R}\times[0,\infty)]$. In order to prove that the class u of $\hat{u}$ is not a C^∞ function, it suffices to show that its associated distribution is not C^∞. From Lemma 3.4 we have that

$$\sup_z\left|\int_0^z [\Theta(x)-\hat{u}_o(\varepsilon, x)]\,dx\right|\to 0 \qquad \text{as} \quad \varepsilon\to 0.$$

Let

$$v(x,t)=\begin{cases}0, & x\leqslant 0\\ x/t, & 0<x<t\\ 1, & x\geqslant t\end{cases}$$

(v is the unique classical solution to (3) with $u_o=\Theta$ satisfying the entropy condition). Applying Proposition 3.1 with $v_i=0$ (which is Lax' result on continuous dependence, see [12, p. 188]), we obtain

$$\sup_{0\leqslant t\leqslant T}|\hat{u}(\varepsilon,\cdot,t)-v(\cdot,t)|_*\to 0 \qquad \text{as} \quad \varepsilon\to 0.$$

Then, in particular, $u\approx v$. ∎

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